Jump process models

Math 622

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1 Motivations

Previously, most of the models of the stock process we have encountered are continuous, i.e the stock price is not supposed to have jumps. Very quickly we see this assumption is restrictive: when the stock pays dividend, the stock price has a downward jump corresponding to the amount of divident payout. However, the dividend payment can be covered within the continuous framework without introducing any new ideas, essentially because the dividend payment times are **deterministic**.

Dividend payments are not the only phenomenons that cause the stock price to have jumps, obviously. In reality, we quickly observe many instances where stock price jumps, and the most important characteristic of these jumps is that they happen at **random** times. Being able to model stock prices that incorporate jumps at random (or more precisly, stopping times) and learning how to price financial products based on these models are the main focus of this Chapter.

2 Some review materials

Reading material: Dan Ocone's Lecture note 1 sections I and II

3 The most basic model of jumping processes: Poisson process

Reading material: Shreve Section 11.2.

3.1 Heuristics about Poisson process

We think of Poisson process as followed: suppose that we have an alarm clock that will ring after a *random* time τ, where τ is exponentially distributed with some mean $\frac{1}{\lambda}$. We keep account of the value of the Poisson process at any time t by the notation $N(t)$. At time 0, we set the alarm clock and set $N(0) = 0$. When the alarm rings, we increase the value of N by 1, that is we set $N(\tau) = 1$ and repeat the whole process (i.e. we reset the alarm clock and increase the value of N by 1 the next time the clock rings). The resulting process $N(t)$ is then a Poisson process with rate λ. We observe that the larger λ is, the clock would be likely to ring sooner and the more jumps would likely happen in a given time interval [0,T]. It is also clear that $N(t)$ is constant in between the "ring" times.

3.2 Formal mathematical definition

a. τ (as a R.V.) is said to be exponentially distributed with rate λ if it has the density

$$f(t) = \lambda e^{-\lambda t}\mathbf{1}_{(t\geq 0)}.$$

It follows that $E(\tau) = \frac{1}{\lambda}$ and $Var(\tau) = \frac{1}{\lambda^2}$. An important property of exponential random variable is the *memoryless property*:

$$\mathbb{P}(\tau > t+s|\tau > s) = \mathbb{P}(\tau > t).$$

b. Let $\tau_i, i = 1, 2, ...$ be a sequence of i.i.d. Exponential(λ). Let $S_k := \sum_{i=1}^{k} \tau_i$. The Poisson process $N(t)$ with rate λ is defined as:

$$N(t) = \sum_{i=1}^{n} \mathbf{1}_{(t\geq S_i)}.$$

τ_i is called the *inter-arrival time.* It is the wait time from the $(i-1)^{th}$ jump to the i^{th} jump. S_i is called the *arrival time.* It is the time of the i^{th} jump.

3.3 Important basic properties

a. Distribution: $N(t)$ is has distribution Poisson(λt), that is

$$\mathbb{P}(N(t) = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}.$$

Proof. Let $S_n = \sum_{i=1}^{n} \tau_i$ be the arrival time, then

$$\begin{aligned}
\mathbb{P}(N(t) = k) &= \mathbb{P}(S_{k+1} > t, S_k \leq t) \\
&= \mathbb{P}(S_{k+1} > t) - \mathbb{P}(S_{k+1} > t, S_k > t) = \mathbb{P}(S_{k+1} > t) - \mathbb{P}(S_k > t).
\end{aligned}$$

From Shreve's Lemma 11.2.1, S_n has Gamma(λ, n) distribution. That is, it has the density:

$$g_n(s) = \frac{(\lambda s)^{n-1}}{(n-1)!} \lambda e^{-\lambda s}, s \geq 0.$$

It is a straight forward matter of integration now to verify that

$$\mathbb{P}(S_{k+1} > t) - \mathbb{P}(S_k > t) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}.$$

The integration can be tedious, however. Another way to verify it is as followed: Denote $f(t) := \mathbb{P}(S_{k+1} > t) - \mathbb{P}(S_k > t)$ and note that $f(t)$ satisfies the following ODE:

$$\begin{aligned}
f'(t) &= g_k(t) - g_{k+1}(t) = \frac{(\lambda t)^{k-1}}{(k-1)!} \lambda e^{-\lambda t} - \frac{(\lambda t)^k}{k!} \lambda e^{-\lambda t} \\
f(0) &= 0.
\end{aligned}$$

It is clear that $f(t) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$ is the *unique* solution to the above ODE. The verification is complete. ■

b. $N(t)$ has independent increment. That is if we denote $\mathcal{F}_t$ to be the filtration generated by $N(s), 0 \leq s \leq t$ then for all $t \leq t_1 < t_2$, $N(t_2) - N(t_1)$ is independent of $\mathcal{F}_t$.

Heuristic reason: Let $0 \leq s < t$. Clearly $N(t) - N(s)$ counts the number of jumps starting from time s. Given all the information up to time s, what is the distribution of the first jump time after s? That is, we want to compute $\mathbb{P}(S_{N(s)+1} \geq t | \mathcal{F}_s)$, where S_n is the arrival time as defined in Shreve (11.2.4). Note that since $N(s)$ represents the number of jumps up to time s, $S_{N(s)+1}$ is exactly the time of *the first jump after time s.*

But this is the same as computing $\mathbb{P}(\tau_{N(s)+1} \geq t - S_{N(s)} | \tau_{N(s)+1} \geq s - S_{N(s)})$. Note that $S_{N(s)}$ here represents the time of the *last jump before time s,* and $\tau_{N(s)+1}$ is *the wait time between the last jump before time s and the first jump after time s.* So $\mathbb{P}(\tau_{N(s)+1} \geq t - S_{N(s)} | \tau_{N(s)+1} \geq s - S_{N(s)})$ asks for the probability that we have to wait until after time t for the first jump after time s, given that we know we have waited up until time s since the last jump before s, which has the same content as $\mathbb{P}(S_{N(s)+1} \geq t|\mathcal{F}_s)$.

Note also that $\mathbb{P}(\tau_{N(s)+1} \geq t - S_{N(s)}|\tau_{N(s)+1} \geq s - S_{N(s)}) = \mathbb{P}(\tau_{N(s)+1} \geq t - s + s - S_{N(s)}|\tau_{N(s)+1} \geq s - S_{N(s)})$. Since $\mathcal{F}_s$ is given, $N(s)$ should be looked at as a constant here. But from the memoryless property of $\tau_{N(s)+1}$, we get

$$\mathbb{P}(\tau_{N(s)+1} \geq t - s + s - \tau_{N(s)}|\tau_{N(s)+1} \geq s - \tau_{N(s)}) = \mathbb{P}(\tau_{N(s)+1} \geq t - s).$$

That is, the first jump time after s can be looked at as an exponential clock starts at time s, hence independent of the past information. Using the independence of inter-arrival times, it is clear now that the increments of $N(t)$ after time s is independent of the information up to time s. ■

c. $N(t)$ has stationary increment. More specifically, $N(t) - N(s)$ has distribution Possion$(\lambda(t - s))$.

Heuristic reason: It follows from the same arguments of part b. ■

4 Generalizations of Poisson process

4.1 Compound Poisson process

Reading material: Shreve 11.3, Ocone's Lecture note 1 section V.D The Poisson process we introduced has the satisfactory property that it jumps at *random* times. However, each of the jump is by definition of length 1, which is rather restrictive. It is desirable in terms of being realistic to have random jumps in our model. To that end, we proceed as followed.

Let $N(t)$ be a Poisson process with rate λ and let $Y_0 = 0, Y_i, i = 1, 2, ...$ be i.i.d.(and also independent of N(t)) with $E(Y_i) = \mu$. Define

$$Q(t) = \sum_{i=0}^{N(t)} Y_i,$$

then $Q(t)$ is called a compound Poisson process. Similar to a Poisson process, $Q(t)$ also has the basic properties of *independent and stationary increments.* We *do not know* the specific distribution of $Q(t) - Q(s)$ (it depends on the distribution of Y_i's , of course), but we do know that $E(Q(t) - Q(s)) = \mu\lambda(t - s)$.

4.2 Pure jump process

Poisson process and compound Poisson process are examples of pure jump processes. See Ocone's lecture note 1 section V.B for discussion.

4.3 Levy process

Reading material: Ocone's lecture note 1 section V.A

So far the three processes that we have encountered in Math Finance: Brownian motion, Poisson and compound Poisson processes have these three properties in common:

- Its value at time 0 is 0 : $X(0) = 0$.
- It has càdlàg path.
- It has stationary and independent increments.

A process $X(t)$ is said to be a Levy process *starting at 0* if it satisfies these three properties (clearly if we change the first property to $X(0) = x$ then we would get a Levy process starting at x). Brownian motion is an example of a continuous Levy process and Poisson process is an example of a pure jump Levy process. Indeed, Brownian motion, compound Poisson process and pure jump process may be thought as "building blocks" of a Levy process (See Levy-Ito decomposition on Wikipedia, for example).

A rather simple but important property of Levy process is as followed: If $X_1, X_2, ...X_n$ are *independent* Levy process then $\sum_{i=1}^{n} X_i$ is a Levy process. In particular, if we consider $S(t) = X(t) + Q(t)$, where $X(t)$ is a Geometric Brownian motion with the drift μ and volatility σ *constant*), $Q(t)$ a compound Poisson process then $S(t)$ is a Levy process.

5 Martingale property

Reading material: Ocone's lecture note 1, section V

In Math finance, we always require the discounted underlying to be a martingale, so that no arbitrage can happen. As mentioned, the Levy process is intimately connected to our stock models, so it's natural to first study the martingale property of Levy processes.

5.1 Levy process

Let $X(t)$ be a Levy process and $\mathcal{F}(t)$ its filtration. If $E(X(1)) = \mu$ then it can be shown that $E(X(t)) = \mu t$. Similarly, if $Var(X(1)) = \sigma^2$ then it can be shown that $Var(X(t)) = \sigma^2 t$. Since X has independent increment, one can check that

$$\begin{aligned} Y(t) &= X(t) - \mu t; \\ Z(t) &= (X(t) - \mu t)^2 - \sigma^2 t \end{aligned}$$

are martingales with respect to $\mathcal{F}(t)$.

5.2 Brownian motion

Let $W(t)$ be a Brownian motion and $\mathcal{F}(t)$ its filtration. Then $W(t)$ and $W^2(t) - t$ are martingales w.r.t. $\mathcal{F}(t)$. More importantly, we have the following exponential martingale associated with Brownian motion:

$$Z(t) = e^{\sigma B_t - \frac{1}{2}\sigma^2 t}.$$

5.3 Poisson process

Let $N(t)$ be a Poisson process and $\mathcal{F}(t)$ its filtration. Then $N(t) - \lambda t$ (called a *compensated Poisson process*) and $(N(t) - \lambda t)^2 - \lambda t$ are martingales w.r.t. $\mathcal{F}(t)$. We also have the following exponential martingale associated with $N(t)$:

$$Z(t) = \exp\big(iuN(t) - \lambda t(e^{iu} - 1)\big), \forall u \in \mathbb{R}.$$

5.4 Compound Poisson process

Let $Q(t)$ be a compound Poisson process and $\mathcal{F}(t)$ its filtration. Let $\sigma^2 = E(Y_i)$. It can be showed that $Var(Q(t)) = \lambda t(\sigma^2 + \mu^2)$. Then $Q(t) - \mu\lambda t$ (called a *compensated compound Poisson process*) and $(Q(t) - \mu\lambda t)^2 - \lambda t(\sigma^2 + \mu^2)$ is a martingale w.r.t. $\mathcal{F}(t)$.

Let $\phi(u) := \mathbb{E}(e^{iuY_1})$ be the characteristic function of Y_i. Then we also have the following exponential martingale associated with $Q(t)$:

$$Z(t) = \exp\big(iuQ(t) - \lambda t(\phi(u) - 1)\big), \forall u \in \mathbb{R}.$$

6 Lebesgue-Stieltjes integral

6.1 Motivation

Now that we have introduced Poisson process, it is easy to see how to incorprate jumps into the current Black-Scholes stock model. Specifically, let $X(t)$ be a geometric Brownian motion:

$$dXt = \mu Xtdt + \sigma XtdWt,$$

and $N(t)$ a Poisson process. Then defining the stock process as $S(t) := X(t) + N(t)$ already gives us a stock price that jumps at random times, and in between the jumps behave as a geometric Brownian motion.

Let Δ_t represent the number of shares of $S(t)$ we hold at time t. As you might remember from the previous material, we need to know how to evaluate the integral $\int_0^t \Delta_s dS_s$, since it is connected with the value of a portfolio that has S as a component. It is reasonable to expect that

$$\int_0^t \Delta_s dS_s = \int_0^t \Delta_s dX_s + \int_0^t \Delta_s dN_s,$$

and we already know how to evaluate $\int_0^t \Delta_s dX_s$ from the chapter on Ito integral. It remains to define $\int_0^t \Delta_s dN_s$.

For each event ω, the path $N_t(\omega)$ (as a function of t) belongs to a special class of functions called *the functions of bounded variation.* For this reason, $\int_0^t \Delta_s dN_s$ is defined via the concept of Lebesgue-Stieltjes integral of classical analysis. It still has some subtleties, however, mostly due to the facts that $N(t)$ has *jumps*, so the regularity (left or right continuity) of the integrand Δ_t affects the value of the integral. For this reason, we will review some basic aspects of the Lebesgue-Stieltjes integral with respect to càdlàg integrator in the next section.

6.2 The Lebesgue-Stieltjes integral

Reading material: Dan Ocone's Lecture note 1, sections III and IV.

Definition 6.1. *We say a function G is of bounded variation if it can be written as a difference of two increasing functions.*

In this section, we will always consider a function G that is càdlàg and of bounded variation defined on $[0, \infty)$.

Definition 6.2. *Let $a > 0$ and let $H(t) = \mathbf{1}_{(0,a]}(t)$. Note that H is left continuous. We define*

$$\int H(t)dG(t) := G(a) - G(0).$$

Definition 6.3. *Let $0 = t_0 < t_1 < ... < t_n < \infty$. Since an integral is linear, for $H(t) = \sum_{i=1}^{n-1} a_i \mathbf{1}_{(t_i,t_{i+1}]}(t)$ we can also define*

$$\int H(t)dG(t) := \sum_{i=0}^{n-1} \int \mathbf{1}_{(t_i,t_{i+1}]}(t)dG(t) := \sum_{i=0}^{n-1} a_i(G(t_{i+1}) - G(t_i)).$$

Remark 6.4. *Let $t > 0$, observe that $\mathbf{1}_{[0,t]}(s)\mathbf{1}_{(t_i,t_{i+1}]}(s) = \mathbf{1}_{(t_i\wedge t,t_{i+1}\wedge t]}(s)$ is also left continuous. Therefore, we will also define*

$$\int_0^t H(s)dG(s) := \int \mathbf{1}_{[0,t]}H(s)dG(s).$$

In particular, for $H(t) = \sum_{i=1}^{n-1} a_i \mathbf{1}_{(t_i,t_{i+1}]}(t)$,

$$\int_0^t H(s)dG(s) := \sum_{i=0}^{n-1} a_i(G(t_{i+1} \wedge t) - G(t_i \wedge t)).$$

The above way to define the integral is of course for a very limited class of integrands. There is an abstract way to define the Lebesgue-Stieltjes integral for a large class of integrands, the so-called measurable functions, that is an extension of the above definition. The precise statement is Theorem 1 in Dan Ocone's note.

For our purpose, most of the time it suffices to know the Lebesgue - Stieltjes integral for a specific form of G. We have three possible cases.

(i) $G(t) = \int_0^t g(s)ds$. Then

$$\int_0^t H(s)dG(s) = \int_0^t H(s)g(s)ds.$$

Let $H(s)$ be a function that only has jump discontinuities. Then the left and right limit of $H(s)$ exists at every point. We define $H(s-)$ to be a function that has the same value as $H(s)$ where $H(s)$ is continuous and takes the left limit value of $H(s)$ where H has jump discontinuities. For example, if $H(s) = \mathbf{1}_{(t\geq a)}(s)$ then $H(s-) = \mathbf{1}_{(t>a)}(s)$. It can also be easily checked that in this case, $H(s-)$ is *left continuous*. We also have,

$$\int_0^t H(s-)dG(s) = \int_0^t H(s-)g(s)ds = \int_0^t H(s)g(s)ds = \int_0^t H(s)dG(s).$$

(ii) Let $0 = t_1 < ... < t_n < \infty$ and $G(t) = \sum_{i=1}^n a_i \mathbf{1}_{(t \geq t_i)}$. Here G is a function of pure jumps. Then

$$\begin{aligned}\int_0^t H(s)dG(s) &= \sum_{0<s\leq t} H(s)\Delta G(s) := \sum_{i,t_i \leq t} H(t_i)(G(t_i) - G(t_i-)) \\ &= \sum_{i,i\leq t} a_i H(t_i).\end{aligned}$$

Remark 6.5. *Note that $Y(t) := \int_0^t H(s)dG(s)$ in this case is also a pure jump function, in particular it is right continuous. $Y(t)$ has the same jump times as $G(t)$, with jump size $\Delta Y(t) = H(t)\Delta G(t)$.*

Remark 6.6. *In this case, $\int_0^t H(s-)dG(s)$ can be different from $\int_0^t H(s)dG(s)$. For example, let $G(s) = \mathbf{1}_{(t\geq a)}, a > 0$. Then $\int_0^a G(s)dG(s) = 1$ while $\int_0^a G(s-)dG(s) = 0$.*

(iii) $G(t) = \int_0^t g(s)ds + J(t)$, where $J(t)$ is a function of pure jumps. Then

$$\int_0^t H(s)dG(s) := \int_0^t H(s)g(s)ds + \sum_{0<s\leq t} H(s)\Delta J(s).$$

7 Stochastic integration w.r.t. semi-martingales

Reading material: Ocone's lecture note 1, section VI, Shreve Section 11.4

7.1 Definition and examples

Let $X(t) = \int_0^t \gamma(s)dW_s + A(t)$, where $W(t)$ is a Brownian motion with respect to a filtration $\mathcal{F}(t)$, $\gamma(t) \in \mathcal{F}(t)$ be such that $\int_0^t \phi(s)dW_s$ is defined and $A(t) \in \mathcal{F}(t)$ a process of bounded variation. $X(t)$ is called a semi-martingale w.r.t. $\mathcal{F}(t)$.

Definition 7.1. *Let $\phi(t) \in \mathcal{F}(t)$ be so that $\int_0^t \phi(s)\gamma(s)dWs$ and $\int_0^t \phi(s)dA(s)$ are defined. Then we define*

$$\int_0^t \phi(s)dX(s) := \int_0^t \phi(s)\gamma(s)dWs + \int_0^t \phi(s)dA(s).$$

It is important to note here that $\int_0^t \phi(s)\gamma(s)dWs$ is an Ito integral, which is not defined path-wise (since $W(t)$ has infinite variation) and $\int_0^t \phi(s)dA(s)$ is a Lebesgue-Stieltjes integral, which is defined pathwise using the definition of Section 4.2.

Example 7.2. *(i) Let $X(t)$ be a compensated compound Poisson process, i.e. $X(t) = Q(t) - \lambda\mu t$ where $Q(t)$ is a compound Poisson process. Let S_k be the jump times of $Q(t)$. Then*

$$\int_0^t \phi(s)dX(s) = \sum_i \phi(S_i)Y_i\mathbf{1}_{(S_i \leq t)} - \int_0^t \lambda\mu\phi(s)ds.$$

(ii) Let $X(t) = W(t) + J(t)$, where $J(t)$ is a pure jump procress. Then

$$\int_0^t \phi(s)dX(s) = \int_0^t \phi(s)dWs + \sum_{0<s\leq t} \phi(s)\Delta J(s).$$

We understand the term $\sum_{0<s\leq t} \phi(s)\Delta J(s)$ as followed: for each event ω, let $0 < t_1(\omega) < t_2(\omega) < ... < t_{n(\omega)}(\omega) \leq t$ be the jump times of $J(t)$. (The fact that there are finitely many jumps in $[0,t]$ and there is no jump at $t = 0$ come from the definition of pure jump process). Also note that the number of jumps in $0,t], n(\omega)$ is random. Then

$$\int_0^t \phi(s)dJ(s)(\omega) = \sum_{0<s\leq t} \phi(s)\Delta J(s) = \sum_{i=1}^{n(\omega)} \phi(t_i)[J(t_i) - J(t_i-)](\omega).$$

7.2 Martingale properties

Reading material Ocone's Lecture note 1, section VII

Suppose we model our stock as

$$S(t) = \sigma W(t) + X(t),$$

where $W(t), X(t) \in \mathcal{F}(t)$ are independent, $W(t)$ is a Brownian motion and $X(t) = Q(t) - \lambda\mu t$ is a compensated compound Poisson process. Then $S(t)$ is a martingale. It is important for us then that if we denote $\phi(t)$ as the number of shares of S we hold at time t , $\int_0^t \phi(r)dS_r$ is a martingale. From Ito integration, we know that if ϕ is an adapted process, then $\int_0^t \phi(s)dW(s)$ is a martingale. So it remains to ask if $\int_0^t \phi(s)dX(s)$ is also a martingale. However, this is not always the case. See Shreve's examples 11.4.4 and 11.4.6.

A suffficient condition for the stochastic integral w.r.t. a jump process (that is also a martingale) to be a martingale is that the integrand is left-continuous (and of course adapted). This is stated in Shreve's theorem 11.4.5. More generally, one can use a predictable integrand (a process that is the limit of a sequence of left-continuous processes) and the stochastic integral w.r.t. a jump martingale will still be a martingale.

In Shreve's example 11.4.6, the following process is considered:

$$\begin{aligned} X(t) &= \int_0^t \mathbf{1}_{[0,S_1]}(s)d(N(s)-\lambda s) \\ &= \int_0^t \mathbf{1}_{[0,S_1]}(s)dN(s) - \int_0^t \mathbf{1}_{[0,S_1]}(s)\lambda ds, \end{aligned}$$

where S_1 is the first jump time of $N(t)$. Note that the integrand here is left continuous.

For $t < S_1$, the integrand $\mathbf{1}_{[0,S_1]}(s) = 0$. Thus $X(t) = -\lambda t$.

For $t = S_1$, $\int_0^t \mathbf{1}_{[0,S_1]}(s)dN(s) = 1\Delta N(S_1) = 1$ while $\int_0^t \mathbf{1}_{[0,S_1]}(s)\lambda ds = \lambda S_1$. Thus $X(t) = 1 - \lambda S_1$.

For $s > S_1, \mathbf{1}_{[0,S_1]}(s) = 0$ thus $X(t) = 1 - \lambda S_1, t \geq S_1$.

We conclude that

$$\begin{aligned} X(t) &= -\lambda t \mathbf{1}_{(t<S_1)} + (1-\lambda S_1)\mathbf{1}_{(t \geq S_1)} \\ &= N(t \wedge S_1) - \lambda(t \wedge S_1). \end{aligned}$$

Here we can use the fact that a stopped martingale is a martingale to conclude that $X(t)$ is a martingale since $N(t) - \lambda t$ is a martingale and the above formula showed that $X(t)$ is a stopped martingale.

Using a similar argument, we have

$$Y(t) = \int_0^t \mathbf{1}_{[0,S_1)}(s)d(N(s)-\lambda s) = -\lambda(t \wedge S_1).$$

Heuristically, $\mathbb{P}(S_1 > 0) = 1$ therefore, for $s < t$,

$$\begin{aligned} \mathbb{P}\big(-\lambda(t \wedge S_1) \leq -\lambda(s \wedge S_1)\big) &= 1; \\ \mathbb{P}\big(-\lambda(t \wedge S_1) < -\lambda(s \wedge S_1)\big) &> 0. \end{aligned}$$

Therefore $\mathbb{E}(-\lambda(t \wedge S_1)) < \mathbb{E}(-\lambda(s \wedge S_1))$ and $Y(t)$ is not a martingale. A rigorous proof is provided in Shreve's.

8 Ito's formula for jump processes

Reading material: Ocone's lecture 1 note section VIII, Shreve's section 11.5

8.1 Ito's formula

The most general jump process we will consider in this chapter has the following form:

$$X(t) = X(0) + \int_0^t \alpha(s)ds + \int_0^t \gamma(s)dWs + J(t),$$

where $J(t)$ is a pure jump process. We also denote by $X^c(t)$ the continuous part of X, that is

$$X^c(t) = X(0) + \int_0^t \alpha(s)ds + \int_0^t \gamma(s)dWs.$$

Given a function $f \in C^2$, we would like to obtain a formula for $df(X(t))$. We have the following observations:

(i) If $X(t) = X^c(t)$, i.e. if X has no jump then we have the classical Ito's formula:

$$df(X(t)) = f'(X(t))dXt + \frac{1}{2}f''(X(t))\gamma^2(t)dt.$$

(ii) If $X(t) = J(t)$, then $f(X(t))$ is also a pure jump process. Moreover,

$$f(X(t)) = f(X(0)) + \sum_{0<s\leq t} f(X(s)) - f(X(s-)).$$

(iii) In general when $X(t) = X^c(t) + J(t)$, intuitively we should have $df(X(t))$ following the classical Ito's formula in between the jumps of X and $\Delta f(X(t)) = f(X(t)) - f(X(t-)$ at the jump points of X.

This leads to the following Ito's formula (see Shreve's theorem 11.5.1)

$$\begin{aligned} f(X(t)) &= f(X(0)) + \int_0^t f'(X(s))dX^c(s) + \int_0^t \frac{1}{2}f''(X(s))\gamma^2(s)ds \\ &\quad + \sum_{0<s\leq t} f(X(s)) - f(X(s-)). \end{aligned}$$

9 Models of stock price with jumps

9.1 Stock models

Recall that before we model the dynamics of a stock $S(t)$ as followed:

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dWt.$$

Also observe that the most important property of $W(t)$ we used in pricing financial models with $S(t)$ is that it is a *martingale*. This motivates us to replace $W(t)$ with a general martingale with jumps. That is, we let

$$M(t) = \int_0^t \alpha(s)ds + \int_0^t \gamma(s)dWs + J(t),$$

where $J(t)$ is a pure jump process be a martingale and consider the following model for $S(t)$:

$$S(t) = S(0) + \int_0^t \mu(s)S(s-)ds + \int_0^t S(s-)dM(s). \tag{1}$$

Intuitively, the reason we use $S(s-)$ in the RHS is so that at the jump of $M(t)$, we have

$$S(t) - S(t-) = S(t-)\Delta J(t). \tag{2}$$

If we think of $\Delta J(t)$ as representing an external shock, then this says the jump in the stock price is its value immediately before the shock occurs multiply with the size of the shock, which makes sense.

Mathematically, using $S(s-)$ in the RHS has the benefit of guaranteeing $\int_0^t S(s-)dM(s)$ to be a martingale under proper conditions (see the discussion in Section 7.2). Either way, it should be noted that we can equivalently write (1) as

$$S(t) = S(0) + \int_0^t (\mu(s) + \alpha(s))S(s)ds + \int_0^t S(s)\gamma(s)dW(s) + \sum_{0<s\leq t} S(s-)\Delta J(s). \tag{3}$$

That is, we only use $S(s-)$ in conjuction with the jumps in $J(s)$.

Relation (5) has another important implication for the jumps of $J(t)$:

$$S(t) = S(t-)(1 + \Delta J(t)).$$

Since we want to use $S(t)$ as a stock price, $S(t) \geq 0$ implies we need to restrict $\Delta J(t) > -1$.

Similar to the classical Black-Scholes model, we have an explicit formula for $S(t)$ satisfying (1) or (3):

$$S(t) = S(0) \exp\Big[\int_0^t [\mu(s) + \alpha(s) - \frac{1}{2}\int_0^t \gamma^2(s)]ds + \int_0^t \gamma(s)dWs\Big] \prod_{0\leq s<t} (1 + \Delta J(s)). \tag{4}$$

Example 9.1. *Geometric Poisson process: If we let* $M(t) = \sigma(N(t) - \lambda t)$ *then*

$$S(t) = S(0) + \int_0^t S(s-)dMs = S(0)e^{-\sigma\lambda t} \prod_{0<s\leq t} (1 + \sigma\Delta N(s)) = S(0)e^{-\sigma\lambda t}(1+\sigma)^{N(t)},$$

since we observe that $1 + \sigma\Delta N(s) = 1 + \sigma$ *at all jump points of* $N(t)$ *and there are exactly* $N(t)$ *jumps at time* t*. Also note that since* σ *is the jump size of the pure jump process* $\sigma N(t)$*, we require* $\sigma > -1$ *as in the discussion above.*

9.2 Some general remarks

Let $W(t)$ be a BM and $N(t)$ be a Poisson process. Observe that

$$X^1(t) = 1 + \int_0^t \sigma X^1(s)ds$$
$$X^2(t) = 1 + \int_0^t \sigma X^2(s)dW(s)$$
$$X^3(t) = 1 + \int_0^t \sigma X^3(s-)dN(s)$$

(note the $X^3(s-)$ in the last equation) have solutions

$$\begin{aligned} X^1(t) &= e^{\sigma t} \\ X^2(t) &= e^{\sigma W(t) - \frac{1}{2}\sigma^2 t} \\ X^3(t) &= (1+\sigma)^{N(t)}, \end{aligned}$$

where the solution for X^1 follows from classical calculus, X^2 from "classical" Ito's formula and X^3 from the calculus for jump processes (see also the discussion about Geometric Poisson process). The point to observe here is that *three very similar differential equations give three distinctly different answers depending on different integrators.*

Also observe that if we apply Ito's formula for jump processes to the $f(N(t)) = (1+\sigma)^{N(t)}$, we get

$$X^3(t) = f(N(t)) = \sum_{s\leq t} (1+\sigma)^{N(s)} - (1+\sigma)^{N(s-)}. \tag{5}$$

This at first glance does not look like the "differential" form

$$\begin{aligned} dX^3(t) &= \sigma X^3(t-)dN(t) \qquad (6) \\ X^3(0) &= 1. \end{aligned}$$

However, we observe from (5) that $\Delta X^3(s) = (1+\sigma)^{N(s)} - (1+\sigma)^{N(s-)}$. Moreover, at the jump point of N

$$\begin{aligned}
\Delta X^3(s) &= (1+\sigma)^{N(s)} - (1+\sigma)^{N(s-)} = (1+\sigma)^{N(s)} - (1+\sigma)^{N(s)-1} \\
&= \sigma(1+\sigma)^{N(s)-1} = \sigma X^3(s-) \\
&= \sigma X^3(s-)\Delta N(s).
\end{aligned}$$

Now the agreement between (5) and (6) are clear. The point here is that it is not immediate to derive "differential" form from the explicit formula of a jump process. Indeed such differential form is not always possible. The fact that $N(t)$ is a counting process (having jump of size 1) is central to the reason why the formula X^3 is nice, as well as that we could re-derive the differential form of $X^3(t)$ from its explicit formula. Replacing $N(t)$ with a general jump process (having arbitrary jump size) in the differential equation for X^3, and you will see that we no longer can easily derive such nice formula anymore.